**Physics 54**

# Electrostatics 3

*Always remember that you're unique, just like everyone else.*
*— Bumper Sticker*

## Storing energy in the E-field

Like other forms of potential energy, electrostatic potential energy is "stored" in the sense that it can be recovered and converted into kinetic energy or other forms.

A device designed specifically to store energy in the electrostatic field is called a **capacitor**, and its effectiveness in storing energy is measured by its **capacitance**. We will analyze capacitors, especially those of simple geometry.

We will find that the energy stored can be thought of as distributed through space, with an amount per unit volume proportional to the squared magnitude of the E-field. This suggests that wherever there is an E-field there is stored energy. This will be an important insight in our later discussion of the electromagnetic field.

## Capacitance

Capacitors store electrostatic energy by separating electric charges. For this purpose they usually consist of a pair of conductors on which equal and opposite charges are placed. The conductors are isolated so that the charges cannot cancel. The resulting E-field causes a difference in potential between the two conductors. Capacitance measures the ratio of the charge to this potential difference:

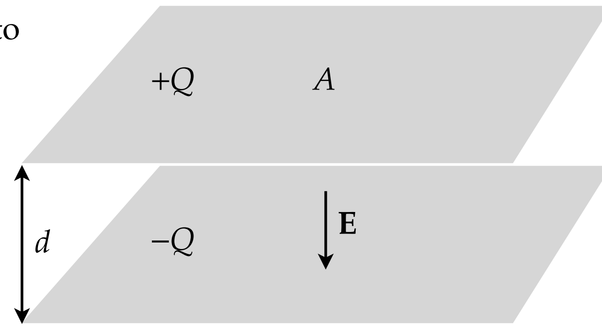
| | |
|---|---|
| Capacitance | Let two conductors carrying charges $+Q$ and $-Q$ be arranged so that the potential difference between them is $\Delta V$. This system has capacitance $C = Q / \Delta V$. |

In fact, $\Delta V$ will be proportional to $Q$, so $C$ actually depends only on the geometric arrangement and the electric properties of any non-conductors that might be present.

One of the simplest geometric arrangements is a pair of flat parallel plate conductors, with dimensions large compared to the spatial separation between them.

We can use the infinite sheet approximation to obtain the E-field from Gauss's law. In that approximation we find that the fields of the oppositely charged plates cancel except in the region between them, where the field is

$$E = 4\pi k\sigma = 4\pi kQ / A \ .$$




This uniform field gives a potential difference

$$\Delta V = Ed = 4\pi kQd / A \ .$$

From the definition of capacitance we thus find a simple formula:

$$\text{Parallel Plate Capacitor: } C = \frac{A}{4\pi kd} = \frac{\varepsilon_0 A}{d}$$

In this analysis we assumed the space between the plates to be empty. We will see later that if there is a non-conducting ("dielectric") material between the plates, the capacitance is increased.

The simple formula above is an approximation, of course, since the E-field near the edges of the plates does not obey the infinite sheet approximation, either in magnitude or direction. But if the separation between the plates is much smaller than their dimensions, this approximation gives useful results.

## Stored energy

To find the stored potential energy we calculate the work done by an external agent to put the charges on the plates.

At time $t$ let the plates have charges $+q$ and $-q$. The potential difference is $\Delta V = q / C$ .

Now let the external agent move charge $dq$ from the negative plate to the positive plate.

Since the charges start and end at rest (no change in kinetic energy), the work done by this agent is the negative of the work done by the electrostatic interaction. It thus *increases* the potential energy by the amount $dU = dq \cdot \Delta V$ , so we have

$$dU = \frac{q}{C}dq$$

Integrating this from $q = 0$ to the final charge $Q$ we have

| | |
|---|---|
| Capacitor energy | $U = \frac{Q^2}{2C} = \frac{1}{2}C \cdot \Delta V^2$ |

The two forms for the energy are both useful; one chooses the form that involves the quantities known in the case at hand.

To gain insight into the nature of this energy, consider the parallel-plate case as an example. Then $\Delta V = Ed$ and $C = \varepsilon_0 A / d$. The stored energy is

$$U = \tfrac{1}{2}(\varepsilon_0 A / d)(Ed)^2 = \tfrac{1}{2}\varepsilon_0 E^2 \cdot Ad$$

Since $Ad$ is the volume of the region between the plates, the last expression on the right has the form of an amount of energy per unit volume multiplied by the volume. We define the **electric energy density** (energy per unit volume) by

| | |
|---|---|
| Electric energy density (free space) | $u_e = \frac{1}{2}\varepsilon_0 E^2$ |

This quantity is a scalar field, giving the energy per unit volume at each point in space. The total stored energy can be found by integrating $u_e$ over all space.

In the parallel plate case, $u_e$ is constant in the region between the plates and zero elsewhere, so we have simply $U = u_e \cdot \text{Volume} = u_e \cdot Ad$.

If there are dielectric materials present the formula for $u_e$ must be altered.

## Dielectrics

Although **dielectrics** do not have charges free to move around, they are not electrically inert. As we have seen, the presence of an external E-field produces aligned dipole moments in such materials, either by aligning permanent moments with the field, or by inducing them, or both.

This "polarization" of the dielectric results in a new E-field created by the dipoles themselves, usually opposite in direction to the applied field. The net field is thus *weaker* than the applied field. The size of the effect is given by a property of the material, called the **dielectric constant**, denoted by $\kappa$ (Greek kappa).

> An external field $E_0$ applied to a material with dielectric constant $\kappa$ will result in a net field in the material given by $E = E_0 / \kappa$.

This simple relationship between the E-fields applies only to materials without crystalline structure, or other asymmetries, that might cause $\kappa$ to be different for different directions of $\mathbf{E_0}$.

If one fills the space between the conductors of a capacitor with dielectric material, the E-field (and hence $\Delta V$) decreases by the factor $1/\kappa$, so the capacitance *increases*.

> If $C_0$ is the capacitance with no dielectric; then with the dielectric $C = \kappa C_0$.

Since the net field $E$ is smaller than $E_0$, $\kappa$ is greater than 1; for empty space $\kappa = 1$.

Suppose we have a capacitor already charged and isolated. Now we slip dielectric material between the plates, filling the gap. The charge $Q$ is unchanged (the capacitor is isolated) but $C$ increases by $\kappa$. The stored energy $U = Q^2 / 2C$ *decreases* by $1/\kappa$.

Since systems go spontaneously to states of lower potential energy, the dielectric will be *pulled into* the gap between the plates by the electrostatic interaction between the charges on the plates and the dipoles in the slab.

The volume of the region between the plates (the only region where $E$ is not negligible) does not change, so the energy density $u_e$ must also decrease by $1/\kappa$. If we write it in terms of the *new* net E-field we find

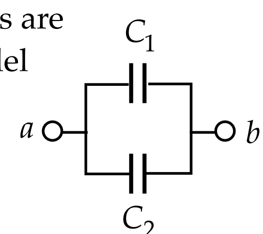
| | |
|---|---|
| Electric energy density (general) | $u_e = \frac{1}{2}\kappa\varepsilon_0 E^2$ |

One must remember that in this formula $E$ is the net field *with the dielectric in place*.

## Capacitors in series and parallel

Capacitors play important roles in electrical circuits, as we will see later. They are connected by conducting wires to other parts of the circuit. If there are two or more capacitors between two given points in the circuit, we can find an equivalent single capacitor that would provide the same capacitance between those points.

Consider the "parallel" combination shown. (In circuit diagrams wires are simple lines and capacitors are represented by a stylized pair of parallel plates. The small circles represent "terminals" at which points the device is connected to other parts of a circuit.)



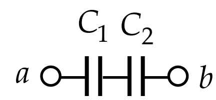

The potential difference between the plates is the same for both capacitors, and the total charge is the sum of their charges. For the effective single capacitor we have $Q = C\,\Delta V$ and for each individually $Q_1 = C_1\,\Delta V$ and $Q_2 = C_2\,\Delta V$. Since $Q = Q_1 + Q_2$, we find from these equations

Parallel Capacitors: $C = C_1 + C_2$

For capacitors in parallel:

- $\Delta V$ is the same for all capacitors;
- The total charge is the sum of the charges on each'
- The total capacitance is the sum of the individual capacitances.

The series case is shown. The charges on the two capacitors are the same (charge is supplied at the terminals from an external source, but the two inner plates continue to have zero total charge). The total potential difference from *a* to *b* is the sum of the individual potential differences. Thus $\Delta V = \Delta V_1 + \Delta V_2$ and $\Delta V = Q / C$, while $\Delta V_1 = Q / C_1$ and $\Delta V_2 = Q / C_2$. These equations lead to




$$\text{Series Capacitors: } \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

For capacitors in series:

- The charge on each is the same;
- The total potential difference is the sum of the individual ones;
- The total capacitance is smaller than any individual one.

Why might one make such combinations in practice? A parallel combination will store more energy for a given potential difference. A series combination divides the potential difference among the capacitors.

Actual capacitors have a maximum voltage rating, giving the largest value of $\Delta V$ that the device can sustain without the material between the plates "breaking down" under the strong E-field and becoming a conductor. A series combination can use capacitors with lower voltage ratings than the total potential difference required.